ITÔ'S FORMULA IN UMD BANACH SPACES AND REGULARITY OF SOLUTIONS OF THE ZAKAI EQUATION

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Abstract. Using the theory of stochastic integration for processes with values in a UMD Banach space developed recently by the authors, an Itô formula is proved which is applied to prove the existence of strong solutions for a class of stochastic evolution equations in UMD Banach spaces. The abstract results are applied to prove regularity in space and time of the solutions of the Zakai equation.

1. Introduction

In this paper we study space-time regularity of strong solutions of the nonautonomous Zakai equation

$$
\begin{aligned}
D_tU(t,x) &= A(t,x,D)U(t,x) + B(x,D)U(t,x)D_tW(t), \quad t\in[0,T], x\in\mathbb{R}^d \\
U(0,x) &= u_0(x), \quad x\in\mathbb{R}^d.
\end{aligned} \tag{1.1}
$$

Here

$$
\begin{aligned}
A(t,x,D) &= \sum_{i,j=1}^d a_{ij}(t,x)D_iD_j + \sum_{i=1}^d q_i(t,x)D_i + r(t,x),\\
B(x,D) &= \sum_{i=1}^d b_i(x)D_i + c(x).
\end{aligned}
$$

This equation arises in filtering theory, and has been studied by many authors, cf. [2, 13, 35] and the references therein. It can be written as an abstract stochastic evolution equation of the form

$$
\begin{aligned}
dU(t) &= A(t)U(t)dt + BU(t)\,dW(t), \qquad t\in[0,T],\\
U(0) &= u_0.
\end{aligned} \tag{1.2}
$$

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Here the linear operators $A(t)$ are closed and densely defined on a suitable Banach space E, the operator B is a generator of a C_0-group on E, and W is a real-valued Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In the framework where E is the Hilbert space $L^2(\mathbb{R}^d)$, the autonomous version of the problem (1.2) has been studied for instance by Da Prato, Iannelli and Tubaro [11] and Da Prato and Zabczyk [13], who proved the existence of strong solutions for this equation. By applying the results to the Zakai equation (1.1) and assuming that $u_0 \in L^2(\mathbb{R}^d)$ almost surely, under suitable regularity conditions on the coefficients the existence of solutions with paths in

$$C([0,T]; L^2(\mathbb{R}^d)) \cap C((0,T]; W^{2,2}(\mathbb{R}^d))$$

is established. If $u_0 \in W^{2,2}(\mathbb{R}^d)$ almost surely, then the solution has paths in $C([0,T]; W^{2,2}(\mathbb{R}^d))$.

In the slightly different setting of a Gelfand triple of separable Hilbert spaces, a class of problems including (1.2) was studied with the same method by Brzeźniak, Capiński and Flandoli [10]. For Zakai's equation they obtain solutions in the space $C([0,T]; L^2(\mathbb{R}^d)) \cap L^2(0,T; W^{1,2}(\mathbb{R}^d))$ for initial values $u_0 \in L^2(\mathbb{R}^d)$.

Using different techniques, Brzeźniak [8] studied a class of equations containing the autonomous case $A(t) \equiv A$ of (1.2) in the setting of martingale type 2 spaces E. For $E = L^p(\mathbb{R}^d)$ with $2 \le p < \infty$ and initial values u_0 taking values almost surely in the Besov space $B^1_{p,2}(\mathbb{R}^d)$, the existence of solutions for the autonomous Zakai equation with paths in $L^2(0,T; W^{2,p}(\mathbb{R}^d))$ and continuous moments in $B^1_{p,2}(\mathbb{R}^d)$ was obtained. The techniques of [11] cannot be extended to the setting of martingale type 2 spaces E, since this would require an extension of the Itô formula for the duality mapping. Here the problem arises that if E has martingale type 2, then E^* has martingale type 2 only if E is isomorphic to a Hilbert space (see [21, 28]).

The method of [11] reduces the stochastic problem (1.2) to a certain deterministic problem. Crucial to this approach is the use of Itô's formula for bilinear forms on Hilbert spaces. This method has been extended by Acquistapace and Terreni [2] to the nonautonomous case using the Kato-Tanabe theory [30, Section 5.3] for operators $A(t)$ with time-dependent domains. In this approach, a technical difficulty arises due to the fact that in the associated deterministic problem, certain operator valued functions are only Hölder continuous, whereas the Kato-Tanabe theory requires their differentiability. This difficulty is overcome by approximation arguments. The authors also note that for the case where the domains $\mathcal{D}(A(t))$ do not depend on time, the methods from [11] can be extended using the Tanabe theory [30, Section 5.2].

In the present paper we will extend the techniques of [11] to UMD spaces E. This class of spaces includes $L^p(\mathbb{R}^d)$ for $p \in (1,\infty)$. The extension relies on the fact that if E is a UMD space, then E^* is a UMD spaces as well. Using the theory of stochastic integration in UMD spaces developed recently in [23], an Itô formula is proved which is subsequently applied to the duality mapping defined on the UMD space $E \times E^*$, $(x, x^*) \mapsto \langle x, x^* \rangle$. For the Zakai equation with initial value $u_0 \in L^p(\mathbb{R}^d)$ almost surely, where $1 < p < \infty$, this results in solutions with paths belonging to

$$C([0,T]; L^p(\mathbb{R}^d)) \cap C((0,T]; W^{2,p}(\mathbb{R}^d)).$$

If $u_0 \in W^{2,p}(\mathbb{R}^d)$ almost surely, the solution has paths in $C([0,T]; W^{2,p}(\mathbb{R}^d))$. For initial values in $L^p(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ (respectively, in $W^{2,p}(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$) for some

$1 \le p < \infty$, the Sobolev embedding theorem then gives solutions with paths in $C((0,T];C^{1,\alpha}(\mathbb{R}^d))$ (respectively, in $C([0,T];C^{1,\alpha}(\mathbb{R}^d)))$ for all $\alpha \in (0,1)$. If u_0 takes its values in a certain interpolation space between $L^p(\mathbb{R}^d)$ and $W^{2,p}(\mathbb{R}^d)$, we obtain that the solution has paths in

$$C([0,T];L^p(\mathbb{R}^d)) \cap L^q(0,T;W^{2,p}(\mathbb{R}^d)),$$

for appropriate $q \in [1,\infty)$.

Rather than using the Kato-Tanabe theory for operators $A(t)$ with time-dependent domains, we shall use the more recent Acquistapace-Terreni theory developed in [3]. The above-mentioned technical difficulty does not occur then.

Another approach was taken by Krylov [19], who developed an L^p-theory for a very general class of time-dependent parabolic stochastic partial differential equations on $\mathbb{R}^d$ by analytic methods. For Zakai's equation with initial conditions u_0 in the Bessel potential space $H^{r+2-\frac{2}{p},p}(\mathbb{R}^d)$, where $r \in \mathbb{R}$ and $2 \le p < \infty$, solutions are obtained with paths in

$$L^p(0,T;H^{r+2,p}(\mathbb{R}^d)).$$

Further L^p-regularity results for the Zakai equation may be found in [18, 20, 26].

2. Itô's formula in UMD Banach spaces

We start with a brief discussion of the L^p-theory of stochastic integration in UMD Banach spaces developed recently in [23]. We fix a separable real Hilbert space H and a real Banach space E, and denote by $\mathcal{L}(H,E)$ the space of all bounded linear operators from H to E.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let F be a Banach space. An F-valued *random variable* is a strongly measurable mapping on Ω into F. The vector space of all F-valued random variables on Ω, identifying random variables if they agree almost surely, is denoted by $L^0(\Omega;F)$. We endow $L^0(\Omega;F)$ with the topology induced by convergence in probability.

An F-valued *process* is a one-parameter family of random variables with values in F. Often we identify a process with the induced mapping $I \times \Omega \to F$, where I is the index set of the process. In most cases below, $I = [0,T]$. A process $\Phi : [0,T] \times \Omega \to \mathcal{L}(H,E)$ will be called *H-strongly measurable* if for all $h \in H$ the process $\Phi h : [0,T] \times \Omega \to E$ defined by $\Phi h(t,\omega) := \Phi(t,\omega)h$, is strongly measurable.

For a separable real Hilbert space $\mathcal{H}$, let $\gamma(\mathcal{H},E)$ denote the operator ideal of γ-radonifying operators in $\mathcal{L}(\mathcal{H},E)$. Recall that $R \in \mathcal{L}(\mathcal{H},E)$ is *γ-radonifying* if for some (equivalently, for each) orthonormal basis $(h_n)_{n\ge1}$ the Gaussian sum $\sum_{n\ge1} \gamma_n R h_n$ converges in $L^2(\Omega;E)$. Here, $(\gamma_n)_{n\ge1}$ is a sequence of independent real-valued standard Gaussian random variables on Ω. We refer to [15, 23, 24, 25] for its definition and relevant properties. Below we shall be interested primarily in the case $\mathcal{H} = L^2(0,T;H)$.

An H-strongly measurable process $\Phi : [0,T] \times \Omega \to \mathcal{L}(H,E)$ is said to *represent* a random variable $X \in L^0(\Omega;\gamma(L^2(0,T;H),E))$ if for all $x^* \in E^*$, for almost all $\omega \in \Omega$ we have $\Phi^*(\cdot,\omega)x^* \in L^2(0,T;H)$ and

$$\langle X(\omega)f, x^*\rangle = \int_0^T [f(t), \Phi^*(t,\omega)x^*]_H \, dt \quad \text{for all } f \in L^2(0,T;H). \tag{2.1}$$

Strong measurability of X can usually be checked with [23, Remark 2.8]. If Φ represents both $X_1, X_2 \in L^0(\Omega;\gamma(L^2(0,T;H),E))$, then $X_1 = X_2$ almost surely by the

Hahn-Banach theorem and the essential separability of the ranges of X_1 and X_2. In the converse direction, if both Φ_1 and Φ_2 represent $X \in L^0(\Omega;\gamma(L^2(0,T;H),E))$, then $\Phi_1 h = \Phi_2 h$ almost everywhere on $\omega \times [0,T]$ for all $h \in H$ (to see this take $f = 1_{[a,b]} \otimes h$ in (2.1); then use the Hahn-Banach theorem and the strong H-measurability of Φ) and therefore $\Phi_1 = \Phi_2$ almost everywhere on $\omega \times [0,T]$. It will often be convenient to identify Φ with X and we will simply write $\Phi \in L^0(\Omega;\gamma(L^2(0,T;H),E))$.

From now on we shall assume that a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ on $(\Omega,\mathcal{F},\mathbb{P})$ is given which satisfies the usual conditions. A process $\Phi : [0,T]\times\Omega \to \mathcal{L}(H,E)$ is called an *elementary process adapted to* $(\mathcal{F}_t)_{t\in[0,T]}$ if it can be written as

$$\Phi(t,\omega) = \sum_{n=0}^{N}\sum_{m=1}^{M} \mathbf{1}_{(t_{n-1},t_n]\times A_{mn}}(t,\omega) \sum_{k=1}^{K} h_k \otimes x_{kmn},$$

where $0 \le t_0 < \cdots < t_N \le T$ and the sets $A_{1n},\dots,A_{Mn} \in \mathcal{F}_{t_{n-1}}$ are disjoint for each n (with the understanding that $(t_{-1},t_0] := \{0\}$ and $\mathcal{F}_{t_{-1}} := \mathcal{F}_0$) and the vectors $h_1,\dots,h_K \in H$ are orthonormal. For such Φ we define the stochastic integral process with respect to W_H as an element of $L^0(\Omega;C([0,T];E))$ as

$$t \mapsto \int_0^t \Phi(t)\,dW_H(t) = \sum_{n=0}^{N}\sum_{m=1}^{M} \mathbf{1}_{A_{mn}}(\omega) \sum_{k=1}^{K} (W_H(t_n\wedge t)h_k - W_H(t_{n-1}\wedge t)h_k)x_{kmn}$$

Here W_H is a cylindrical Brownian motion. For a process $\Phi : [0,T]\times\Omega \to \mathcal{L}(H,E)$ we say that Φ is *scalarly in* $L^0(\Omega;L^2(0,T;H))$ if for all $x^*\in E^*$, for almost all $\omega\in\Omega$ we have $\Phi^*(\cdot,\omega)x^* \in L^2(0,T;H)$. The following result from [23] extends the integral to a larger class of processes.

Proposition 2.1. *Assume that E is a UMD space and let W_H be an H-cylindrical Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$. For an H-strongly measurable and adapted process $\Phi : [0,T]\times\Omega \to \mathcal{L}(H,E)$ which is scalarly in $L^0(\Omega;L^2(0,T;H))$ the following assertions are equivalent:*

(1) *there exist elementary adapted processes $\Phi_n : [0,T]\times\Omega \to \mathcal{L}(H,E)$ such that:*
   (i) *for all $h\in H$ and $x^*\in E^*$,*
   $$\langle \Phi h, x^*\rangle = \lim_{n\to\infty}\langle \Phi_n h, x^*\rangle \quad \textit{in measure};$$
   (ii) *there exists a process $\zeta \in L^0(\Omega;C([0,T];E))$ such that*
   $$\zeta = \lim_{n\to\infty}\int_0^{\cdot} \Phi_n(t)\,dW_H(t) \quad \textit{in } L^0(\Omega;C([0,T];E)).$$
(2) *There exists a process $\zeta \in L^0(\Omega;C([0,T];E))$ such that for all $x^*\in E^*$,*
$$\langle \zeta, x^*\rangle = \int_0^{\cdot} \Phi^*(t)x^*\,dW_H(t) \quad \textit{in } L^0(\Omega;C[0,T]).$$
(3) $\Phi \in L^0(\Omega;\gamma(L^2(0,T;H),E))$.

The processes ζ in (1) and (2) are indistinguishable and it is uniquely determined as an element of $L^0(\Omega;C([0,T];E))$. It is a continuous local martingale starting at 0, and for all $p\in(1,\infty)$ there exists a constant $0 < C_{p,E} < \infty$ such that

$$C_{p,E}^{-1}\mathbb{E}\|\Phi\|^p_{\gamma(L^2(0,T;H),E)} \le \mathbb{E}\sup_{t\in[0,T]}\|\zeta(t)\|^p \le C_{p,E}\mathbb{E}\|\Phi\|^p_{\gamma(L^2(0,T;H),E)}.$$

A process $\Phi : [0,T] \times \Omega \to \mathcal{L}(H,E)$ satisfying the equivalent conditions of the theorem will be called *stochastically integrable* with respect to W_H. The process ζ is called the *stochastic integral process* of Φ with respect to W_H, notation

$$\zeta = \int_0^\cdot \Phi(t)\, dW_H(t).$$

The following lemma will be needed in Section 3 and shows that condition (2) in Proposition 2.1 can be weakened.

Lemma 2.2. *Let E be a UMD Banach space and let F be a dense subspace of E^*. Let $\Phi : [0,T] \times \Omega \to \mathcal{L}(H,E)$ be an H-strongly measurable and adapted process such that for all $x^* \in F$, $\Phi^* x^* \in L^2(0,T;H)$ almost surely. If there exists process $\zeta \in L^0(\Omega; C([0,T];E))$ such that for all $x^* \in F$ we have*

$$\langle \zeta, x^* \rangle = \int_0^\cdot \Phi^*(s) x^*\, dW_H(s) \quad \text{in } L^0(\Omega; C[0,T]), \tag{2.2}$$

then Φ is stochastically integrable with respect to W_H and

$$\zeta = \int_0^\cdot \Phi(s)\, dW_H(s) \quad \text{in } L^0(\Omega; C([0,T];E)).$$

Proof. By Proposition 2.1, it suffices to show that $\Phi^* x^* \in L^2(0,T;H)$ almost surely and that(2.2) holds for all $x^* \in E^*$. To do so, fix $x^* \in E^*$ arbitrary and choose elements $x_n^* \in F$ such that $x^* = \lim_{n\to\infty} x_n^*$ in E^*. Clearly we have $\langle \zeta, x^* \rangle = \lim_{n\to\infty} \langle \zeta, x_n^* \rangle$ in $L^0(\Omega; C[0,T])$. An application of [17, Proposition 17.6] shows that the processes $\Phi^* x_n^*$ define a Cauchy sequence in $L^0(\Omega; L^2(0,T;H))$. By a standard argument we obtain that $\Phi^* x^* \in L^0(\Omega; L^2(0,T;H))$ and $\lim_{n\to\infty} \Phi^* x_n^* = \Phi^* x^*$ in $L^0(\Omega; L^2(0,T;H))$. By another application of [17, Proposition 17.6] we conclude that

$$\int_0^\cdot \Phi^*(s) x^*\, dW_H(s) = \lim_{n\to\infty} \int_0^\cdot \Phi^*(s) x_n^*\, dW_H(s) = \lim_{n\to\infty} \langle \zeta, x_n^* \rangle = \langle \zeta, x^* \rangle$$

in $L^0(\Omega; C[0,T])$. □

The next lemma defines a trace which will be needed in the statement of the Itô formula.

Lemma 2.3. *Let E, F, G be Banach spaces and let $(h_n)_{n\geq 1}$ be an orthonormal basis of H. Let $R \in \gamma(H,E)$, $S \in \gamma(H,F)$ and $T \in \mathcal{L}(E, \mathcal{L}(F,G))$ be given. Then the sum*

$$\mathrm{Tr}_{R,S} T := \sum_{n\geq 1} (TRh_n)(Sh_n) \tag{2.3}$$

converges in G and does not depend on the choice of the orthonormal basis. Moreover,

$$\|\mathrm{Tr}_{R,S} T\| \leq \|T\| \|R\|_{\gamma(H,E)} \|S\|_{\gamma(H,F)}. \tag{2.4}$$

If $E = F$ we shall write $\mathrm{Tr}_R := \mathrm{Tr}_{R,R}$.

Proof. First assume that $S = \sum_{n=1}^N h_n \otimes y_n$ with $y_1, \ldots, y_N \in F$. Then the convergence of the series in (2.3) is obvious. Letting $\xi_R = \sum_{n=1}^N \gamma_n R h_n$ and $\xi_S = \sum_{n=1}^N \gamma_n S h_n$ we obtain

$$\|\mathrm{Tr}_{R,S} T\| = \|\mathbb{E} T(\xi_R)(\xi_S)\| \leq \|T\| (\mathbb{E}\|\xi_R\|^2)^{\frac{1}{2}} (\mathbb{E}\|\xi_S\|^2)^{\frac{1}{2}} \leq \|T\| \|R\|_{\gamma(H,E)} \|S\|_{\gamma(H,F)}.$$

Now let $S \in \gamma(H,F)$ be arbitrary. For each $N \geq 1$, let $P_N \in \mathcal{L}(H)$ denote the orthogonal projection on $\mathrm{span}\{h_n : n \leq N\}$. Letting $S_n = S \circ P_n$, we have $S = \lim_{n\to\infty} S_n$ in $\gamma(H,F)$. For all $m, n \geq 1$, we have

$$\|\mathrm{Tr}_{R,S_n} T - \mathrm{Tr}_{R,S_m} T\| = \|\mathrm{Tr}_{R,S_n - S_m} T\| \leq \|T\| \|R\|_{\gamma(H,E)} \|S_n - S_m\|_{\gamma(H,F)}.$$

Therefore, $(\mathrm{Tr}_{R,S_n} T)_{n\geq 1}$ is a Cauchy sequence in G, and it converges. Clearly, for all $N \geq 1$, $\mathrm{Tr}_{R,S_N} T = \sum_{n=1}^{N} (TRh_n)(Sh_n)$. Now the convergence of (2.3) and the estimate (2.4) follow.

Next we show that the trace is independent of the choice of the orthonormal basis $(h_n)_{n\geq 1}$. Let $(e_n)_{n\geq 1}$ be another orthonormal basis for H. For $R = \sum_{m=1}^{M} h_m \otimes x_m$ with $x_1, \dots, x_M \in E$ and $S = \sum_{n=1}^{N} h_n \otimes y_n$ with $y_1, \dots, y_N \in F$, we have

$$\begin{aligned}
\sum_{k\geq 1} T(Re_k)(Se_k) &= \sum_{k\geq 1} \sum_{m\geq 1} \sum_{n\geq 1} [e_k, h_m][e_k, h_n] T(Rh_m)(Sh_n) \\
&= \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{k\geq 1} [e_k, h_m][e_k, h_n] T(Rh_m)(Sh_n) \\
&= \sum_{m=1}^{M} \sum_{n=1}^{N} \delta_{mn} T(Rh_m)(Sh_n) = \mathrm{Tr}_{R,S} T.
\end{aligned}$$

The general case follows from an approximation argument as before. $\square$

A function $f : [0,T] \times E \to F$ is said to be of *class* $C^{1,2}$ if f is differentiable in the first variable and twice Fréchet differentiable in the second variable and the functions f, $D_1 f$, $D_2 f$ and $D_2^2 f$ are continuous on $[0,T] \times E$. Here $D_1 f$ and $D_2 f$ are the derivatives with respect to the first and second variable, respectively. We proceed with a version of Itô's formula as announced in [23].

Theorem 2.4 (Itô formula)**.** *Let E and F be UMD spaces. Assume that $f : [0,T] \times E \to F$ is of class $C^{1,2}$. Let $\Phi : [0,T] \times \Omega \to \mathcal{L}(H,E)$ be an H-strongly measurable and adapted process which is stochastically integrable with respect to W_H and assume that the paths of Φ belong to $L^2(0,T;\gamma(H,E))$ almost surely. Let $\psi : [0,T] \times \Omega \to E$ be strongly measurable and adapted with paths in $L^1(0,T;E)$ almost surely. Let $\xi : \Omega \to E$ be strongly $\mathcal{F}_0$-measurable. Define $\zeta : [0,T] \times \Omega \to E$ by*

$$\zeta = \xi + \int_0^{\cdot} \psi(s)\, ds + \int_0^{\cdot} \Phi(s)\, dW_H(s).$$

Then $s \mapsto D_2 f(s, \zeta(s))\Phi(s)$ is stochastically integrable and almost surely we have, for all $t \in [0,T]$,

$$\begin{aligned}
f(t,\zeta(t)) - f(0,\zeta(0)) &= \int_0^t D_1 f(s,\zeta(s))\, ds + \int_0^t D_2 f(s,\zeta(s))\psi(s)\, ds \\
&\quad + \int_0^t D_2 f(s,\zeta(s))\Phi(s)\, dW_H(s) \\
&\quad + \tfrac{1}{2} \int_0^t \mathrm{Tr}_{\Phi(s)}\big(D_2^2 f(s,\zeta(s))\big)\, ds.
\end{aligned} \tag{2.5}$$

The first two integrals and the last integral are almost surely defined as a Bochner integral. To see this for the last integral, notice that by Lemma 2.3 we have

$$\begin{aligned}\int_0^t \left\|\mathrm{Tr}_{\Phi(s)}\big(D_2^2 f(s,\zeta(s))\big)\right\| ds &\le \int_0^t \|D_2^2 f(s,\zeta(s))\| \|\Phi(s)\|^2_{\gamma(H,E)}\, ds \\ &\le \sup_{s\in[0,T]} \|D_2^2 f(s,\zeta(s))\| \|\Phi\|^2_{L^2(0,T;\gamma(H,E))}\end{aligned}$$

almost surely.

Remark 2.5. In the situation of Theorem 2.4, Via Proposition 2.1, the stochastic integrability implies that $\Phi \in L^0(\Omega;\gamma(L^2(0,T;H),E))$. If, in addition to the assumptions of Theorem 2.4, we assume that E has type 2, then

$$L^2(0,T;\gamma(H,E)) \hookrightarrow \gamma(L^2(0,T;H),E)$$

canonically. Therefore, the assumption that Φ is stochastically integrable is automatically fulfilled since $\Phi \in L^0(\Omega;L^2(0,T;\gamma(H,E)))$. In that case the theorem reduces to the Itô formula in [9, 25].

If E has cotype 2, then

$$\gamma(L^2(0,T;H),E) \hookrightarrow L^2(0,T;\gamma(H,E))$$

canonically and the assumption that $\Phi \in L^0(\Omega;L^2(0,T;\gamma(H,E)))$ is automatically fulfilled if Φ is stochastically integrable.

As a consequence of Theorem 2.4 we obtain the following corollary.

Corollary 2.6. *Let E_1, E_2 and F be UMD Banach spaces and let $f : E_1 \times E_2 \to F$ be a bilinear map. Let $(h_n)_{n\ge1}$ be an orthonormal basis of H. For $i = 1,2$ let $\Phi_i : [0,T]\times\Omega \to \mathcal{L}(H,E_i)$, $\psi_i : [0,T]\times\Omega \to E$ and $\xi_i : \Omega \to E_i$ satisfy the assumptions of Theorem 2.4 and define*

$$\zeta_i(t) = \xi_i + \int_0^t \psi_i(s)\, ds + \int_0^t \Phi_i(s)\, dW_H(s).$$

Then, almost surely for all $t\in[0,T]$,

$$\begin{aligned}f(\zeta_1(t),\zeta_2(t)) - f(\zeta_1(0),\zeta_2(0)) &= \int_0^t f(\zeta_1(s),\psi_2(s)) + f(\psi_1(s),\zeta_2(s))\, ds \\ &\quad + \int_0^t f(\zeta_1(s),\Phi_2(s)) + f(\Phi_1(s),\zeta_2(s))\, dW_H(s) \\ &\quad + \int_0^t \sum_{n\ge1} f(\Phi_1(s)h_n, \Phi_2(s)h_n)\, ds.\end{aligned}$$

In particular, for a UMD space E, taking $E_1 = E$, $E_2 = E^*$, $F = \mathbb{R}$ and $f(x,x^*) = \langle x, x^*\rangle$, it follows that almost surely for all $t\in[0,T]$,

(2.6)

$$\begin{aligned}\langle \zeta_1(t),\zeta_2(t)\rangle - \langle \zeta_1(0),\zeta_2(0)\rangle &= \int_0^t \langle \zeta_1(s),\psi_2(s)\rangle + \langle \psi_1(s),\zeta_2(s)\rangle\, ds \\ &\quad + \int_0^t \langle \zeta_1(s),\Phi_2(s)\rangle + \langle \Phi_1(s),\zeta_2(s)\rangle\, dW_H(s) \\ &\quad + \int_0^t \sum_{n\ge1} \langle \Phi_1(s)h_n, \Phi_2(s)h_n\rangle\, ds.\end{aligned}$$

The result of Corollary 2.6 for martingale type 2 spaces E_1, E_2 and F can be found in [9, Corollary 2.1]. However, we want to emphasize that it is not possible to obtain (2.6) with martingale type 2 methods, since E and E^* have martingale type 2 if and only if E is isomorphic to a Hilbert space.

For the proof of theorem 2.4 we need two lemmas.

Lemma 2.7. *Let E be a UMD space. Let $\Phi : [0,T]\times\Omega \to \mathcal{L}(H,E)$ be an H-strongly measurable and adapted process which is stochastically integrable with respect to W_H and assume that its paths belong to $L^2(0,T;\gamma(H,E))$ almost surely. Then there exists a sequence of elementary adapted processes $(\Phi_n)_{n\geq 1}$ such that*

$$\Phi = \lim_{n\to\infty} \Phi_n \text{ in } L^0(\Omega; L^2(0,T;\gamma(H,E))) \cap L^0(\Omega;\gamma(L^2(0,T;H),E)).$$

Proof. Let $(h_n)_{n\geq 1}$ be an orthonormal basis for H and denote by P_n the projection onto the span of $\{h_1,\dots,h_n\}$ in H. Define $\Psi_n : [0,T]\times\Omega \to \gamma(H,E)$ as

$$\begin{aligned}\Psi_n(t,\omega)h :&= \mathbb{E}(R^{\delta_n}(\Phi(\cdot,\omega)P_n h)|\mathcal{G}_n)(t)\\ &= \sum_{k=1}^{2^n} \mathbf{1}_{((k-1)2^{-n}T,k2^{-n}T]}(t)\int_{(k-2)2^{-n}T}^{(k-1)2^{-n}T} \Phi(s)P_n h\, ds,\end{aligned}$$

where $R^{\delta_n} : L^2(0,T;E) \to L^2(0,T;E)$ denotes right translation over $\delta_n = 2^{-n}$, $\mathcal{G}_n$ is the n-th dyadic σ-algebra. By [23, Proposition 2.1], $\Phi = \lim_{n\to\infty}\Psi_n$ in $L^0(\Omega;L^2(0,T;\gamma(H,E)))$ and $\Phi = \lim_{n\to\infty}\Psi_n$ in $L^0(\Omega;\gamma(L^2(0,T;H),E))$.

The processes Ψ_n are not elementary in general, but of the form

$$\Psi_n = \sum_{k=1}^{K_n} 1_{(t_{kn},t_{k+1,n}]} \otimes \sum_{i=1}^{n} h_i \otimes \xi_{ikn},$$

where each ξ_{ikn} is an $\mathcal{F}_{t_{kn}}$-measurable E-valued random variable. Approximating each ξ_{ikn} in probability by a sequence of $\mathcal{F}_{t_{kn}}$-simple random variables we obtain a sequence of elementary adapted processes $(\Psi_{nm})_{m\geq 1}$ such that $\lim_{m\to\infty}\Psi_{nm} = \Psi_n$ in $L^0(\Omega;L^2(0,T;\gamma(H,E)))$ and $\lim_{m\to\infty}\Psi_{nm} = \Psi_n$ in $L^0(\Omega;\gamma(L^2(0,T;H),E))$. For an appropriate subsequence $(m_n)_{n\geq 1}$, the elementary adapted processes Φ_{nm_n} have the required properties. □

The next lemma is proved in a similar way.

Lemma 2.8. *Let E be a Banach space, and let $\psi \in L^0(\Omega;L^1(0,T;E))$ be an adapted process. Then there exists a sequence of elementary adapted processes $(\psi_n)_{n\geq 1}$ such that $\psi = \lim_{n\to\infty}\psi_n$ in $L^0(\Omega;L^1(0,T;E))$.*

Proof of Theorem 2.4. The proof is divided into several steps.

Step 1 – Reduction to the case $F=\mathbb{R}$. Assume the theorem holds in the case $F=\mathbb{R}$. Applying this to $\langle f, x^*\rangle$ for $x^*\in E^*$ arbitrary we obtain

$$\begin{aligned}\langle f(t,\zeta(t)),x^*\rangle - \langle f(0,\zeta(0)),x^*\rangle = &\Big\langle \int_0^t D_1 f(s,\zeta(s))\,ds, x^*\Big\rangle \\ &+ \Big\langle \int_0^t D_2 f(s,\zeta(s))\psi(s)\,ds, x^*\Big\rangle \\ &+ \int_0^t \big(D_2 f(s,\zeta(s))\Phi(s)\big)^* x^*\,dW_H(s) \\ &+ \frac{1}{2}\Big\langle \int_0^t \mathrm{Tr}\big(D_2^2 f(s,\zeta(s))(\Phi(s),\Phi(s))\big)\,ds, x^*\Big\rangle.\end{aligned}$$

An application of Proposition 2.1 (2) to the pathwise continuous process

$$\begin{aligned}f(\cdot,\zeta) - \langle f(0,\zeta(0)) - \int_0^{\cdot} D_1 f(s,\zeta(s))\,ds - \int_0^{\cdot} D_2 f(s,\zeta(s))\psi(s)\,ds \\ - \frac{1}{2}\int_0^{\cdot} \mathrm{Tr}\big(D_2^2 f(s,\zeta(s))(\Phi(s),\Phi(s))\big)\,ds.\end{aligned}$$

shows that $D_2 f(\cdot,\zeta)\Phi$ is stochastically integrable and (2.5) holds. It follows that it suffices to consider $F=\mathbb{R}$.

Step 2 – Reduction to elementary adapted processes. Assume the theorem holds for elementary processes. By path continuity it suffices to show that for all $t\in[0,T]$ almost surely (2.5) holds. Define the sequence $(\zeta_n)_{n\geq 1}$ in $L^0(\Omega;C([0,T];E))$ by

$$\zeta_n(t) = \xi_n + \int_0^t \psi_n(s)\,ds + \int_0^t \Phi_n(s)\,dW_H(s),$$

where $(\xi_n)_{n\geq1}$ is a sequence of $\mathcal{F}_0$-measurable simple functions with $\xi=\lim_{n\to\infty}\xi_n$ almost surely and $(\Phi_n)_{n\geq1}$ and $(\psi_n)_{n\geq1}$ are chosen from Lemma 2.7 and 2.8. By [23, Theorems 5.5 and 5.9] we have $\zeta=\lim_{n\to\infty}\zeta_n$ in $L^0(\Omega;C([0,T];E))$. We may choose $\Omega_0\subseteq\Omega$ of full measure and a subsequence which we again denote by $(\zeta_n)_{n\geq1}$ such that

$$\zeta = \lim_{n\to\infty}\zeta_n \text{ in } C([0,T];E) \text{ on } \Omega_0. \tag{2.7}$$

Thus, in order to prove (2.5) holds for the triple (ξ,ψ,Φ) it suffices to show that all terms in (2.5) depend continuously on (ξ,ψ,Φ). This is standard, but we include the details for convenience.

For the left hand side of (2.5) it follows from (2.7) that

$$\lim_{n\to\infty} f(t,\zeta_n(t)) - f(0,\zeta_n(0)) = f(t,\zeta(t)) - f(0,\zeta(0)) \text{ almost surely.}$$

For a continuous function $p:[0,T]\times E\to B$, where B is some Banach space, and $\omega\in\Omega_0$ fixed the set

$$\{p(s,\zeta_n(s,\omega)):\ s\in[0,T],\ n\geq1\}\cup\{p(s,\zeta(s,\omega)):\ s\in[0,T]\}$$

is compact in B, hence bounded. Let $K=K(\omega)$ denote the maximum of these bounds obtained by applying this to the functions f, $D_2 f$ and $D_2^2 f$. By Lemma 2.8, (2.7) and dominated convergence, on Ω_0 we obtain

$$\lim_{n\to\infty}\int_0^t D_1 f(s,\zeta_n(s))\,ds = \int_0^t D_1 f(s,\zeta(s))\,ds,$$

$$\lim_{n\to\infty}\int_0^t D_2 f(s,\zeta_n(s))\psi_n(s)\,ds = \int_0^t D_2 f(s,\zeta(s))\psi(s)\,ds.$$

For the stochastic integral term in (2.5), by [17, Lemma 17.12] it is enough to show that on Ω_0,

$$\lim_{n\to\infty}\|D_2 f(\cdot,\zeta)\Phi - D_2 f(\cdot,\zeta_n)\Phi_n\|_{L^2(0,T;H)} = 0. \tag{2.8}$$

Here $D_2 f(\cdot,\zeta)$ and $D_2 f(\cdot,\zeta_n)$ stand for $D_2 f(\cdot,\zeta(\cdot))$ and $D_2 f(\cdot,\zeta_n(\cdot))$, respectively. But, by Lemma 2.7 we have

$$\begin{aligned}\lim_{n\to\infty}\|D_2 f(\cdot,\zeta_n)(\Phi-\Phi_n)\|_{L^2(0,T;H)} &\le K\lim_{n\to\infty}\|\Phi-\Phi_n\|_{L^2(0,T;\mathcal{L}(H,E))}\\ &\le K\lim_{n\to\infty}\|\Phi-\Phi_n\|_{L^2(0,T;\gamma(H,E))} = 0,\end{aligned}$$

and, by (2.7) and dominated convergence,

$$\lim_{n\to\infty}\|(D_2 f(\cdot,\zeta) - D_2 f(\cdot,\zeta_n))\Phi\|_{L^2(0,T;H)} = 0$$

on Ω_0. Together these estimates give (2.8).

For the last term in (2.5) we have

$$\begin{aligned}&\|\mathrm{Tr}_\Phi(D_2^2 f(\cdot,\zeta)) - \mathrm{Tr}_{\Phi_n}(D_2^2 f(\cdot,\zeta_n))\|_{L^1(0,T)}\\ &\qquad\le \|\mathrm{Tr}_\Phi(D_2^2 f(\cdot,\zeta)) - \mathrm{Tr}_\Phi(D_2^2 f(\cdot,\zeta_n))\|_{L^1(0,T)}\\ &\qquad\quad + \|\mathrm{Tr}_\Phi(D_2^2 f(\cdot,\zeta_n)) - \mathrm{Tr}_{\Phi_n}(D_2^2 f(\cdot,\zeta_n))\|_{L^1(0,T)}.\end{aligned}$$

The first term tends to 0 on Ω_0 by Lemma 2.3, (2.7) and dominated convergence. For the second term, by Lemma 2.3, the Cauchy-Schwartz inequality and Lemma 2.7 we have

$$\begin{aligned}&\|\mathrm{Tr}_\Phi(D_2^2 f(\cdot,\zeta_n)) - \mathrm{Tr}_{\Phi_n}(D_2^2 f(\cdot,\zeta_n))\|_{L^1(0,T)}\\ &\qquad\le \|\mathrm{Tr}_\Phi(D_2^2 f(\cdot,\zeta_n)) - \mathrm{Tr}_{\Phi,\Phi_n}(D_2^2 f(\cdot,\zeta_n))\|_{L^1(0,T)}\\ &\qquad\quad + \|\mathrm{Tr}_{\Phi,\Phi_n}((D_2^2 f(\cdot,\zeta_n)) - \mathrm{Tr}_{\Phi_n}(D_2^2 f(\cdot,\zeta_n))\|_{L^1(0,T)}\\ &\qquad\le K\|\Phi\|_{L^2(0,T;\gamma(H,E))}\|\Phi-\Phi_n\|_{L^2(0,T;\gamma(H,E))}\\ &\qquad\quad + K\|\Phi_n\|_{L^2(0,T;\gamma(H,E))}\|\Phi-\Phi_n\|_{L^2(0,T;\gamma(H,E))},\end{aligned}$$

which tends to 0 on Ω_0 as well.

Step 3 – If ξ is simple, ψ and Φ are elementary, they take their values in a finite dimensional subspace $E_0 \subseteq E$ and there exists a finite dimensional subspace H_0 of H such that $H = H_0 \oplus \mathrm{Ker}(\Phi)$. Since E_0 is isomorphic to some $\mathbb{R}^n$ and H_0 is isomorphic to some $\mathbb{R}^m$, (2.5) follows from the corresponding real valued Itô formula. □

Remark 2.9. With more elaborate methods one may show that in Corollary 2.6 the assumption $\Phi \in L^0(\Omega; L^2(0,T;\gamma(H,E_i)))$ is not needed. The proof of this result depends heavily on the fact that $D^2 f$ is constant in that case. For general functions f of class $C^{1,2}$ we do not know if the assumption can be omitted.

3. The abstract stochastic equation

After these preparations we start our study of the problem

$$\begin{aligned}dU(t) &= A(t)U(t)dt + \sum_{n=1}^N B_n U(t)dW_n(t), \quad t\in[0,T],\\ U(0) &= u_0.\end{aligned} \tag{3.1}$$

The processes $W_n = (W_n(t))_{t\in[0,T]}$ are independent standard Brownian motions on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and are adapted to some filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. The initial random variable $u_0 : \Omega \to E$ is assumed to be strongly $\mathcal{F}_0$-measurable. Concerning the operators $A(t) : \mathcal{D}(A(t)) \subseteq E \to E$ and $B_n : \mathcal{D}(B_n) \subseteq E \to E$ we assume the following hypotheses.

(H1) The operators $A(t)$ are closed and densely defined;
(H2) The operators B_n generate commuting C_0-groups $G_n = (G_n(t))_{t\in\mathbb{R}}$ on E;
(H3) For all $t \in [0,T]$ we have $\mathcal{D}(A(t)) \subseteq \bigcap_{n=1}^N \mathcal{D}(B_n^2)$.

Defining $\mathcal{D}(C(t)) := \mathcal{D}(A(t))$ and $C(t) := A(t) - \frac{1}{2}\sum_{n=1}^N B_n^2$, we further assume

(H4) There exists a $\lambda \in \mathbb{R}$ with $\lambda \in \varrho(A(t)) \cap \varrho(C(t))$ for all $t \in [0,T]$, such that the functions $t \mapsto B_n^2 R(\lambda, A(t))$ and $t \mapsto B_n^2 R(\lambda, C(t))$ are strongly continuous on $[0,T]$.

Hypothesis (H4) is automatically fulfilled in the case $A(t)$ is independent of t. Below it is showed that it is fulfilled in several time dependent situation as well.

An E-valued process $U = \{U(t)\}_{t\in[0,T]}$ is called a *strong solution of* (3.1) *on the interval* $(0,T]$ if $U \in C([0,T];E)$ almost surely, $U(0) = u_0$, and for all $\varepsilon > 0$ the following conditions are satisfied:

(1) For almost all $\omega \in \Omega$, $U(t,\omega) \in \mathcal{D}(A(t))$ for almost all $t \in [\varepsilon, T]$ and the path $t \mapsto A(t)U(t,\omega)$ belongs to $L^1(\varepsilon, T; E)$;
(2) For $n = 1, \dots, N$ the process $B_n U$ is stochastically integrable with respect to W_n on $[\varepsilon, T]$;
(3) Almost surely,

$$U(t) = U(\varepsilon) + \int_\varepsilon^t A(s)U(s)\,ds + \sum_{n=1}^N \int_\varepsilon^t B_n U(s)\,dW_n(s) \quad \text{for all } t \in [\varepsilon, T].$$

Note that by path continuity, the exceptional sets may be chosen independently of $\varepsilon \in (0,T]$. We call U a *strong solution on the interval* $[0,T]$ if U satisfies (1), (2) and (3) with $\varepsilon = 0$.

Assuming Hypotheses (H1)–(H4), in the Hilbert space setting the existence of strong solutions has been established in [11] (see also [13, Section 6.5]) by reducing the stochastic problem to a deterministic one and then solving the latter by parabolic methods. Here we shall extend this method to the setting of UMD spaces using the bilinear Itô formula of the previous section.

Define $G : \mathbb{R}^N \to \mathcal{L}(E)$ as

$$G(a) := \prod_{n=1}^N G_n(a_n).$$

Note that each $G(a)$ is invertible with inverse $G^{-1}(a) := (G(a))^{-1} = G(-a)$. For $t \in [0,T]$ and $\omega \in \Omega$ we define the operators $C_W(t,\omega) : \mathcal{D}(C_W(t,\omega)) \subseteq E \to E$ by

$$\mathcal{D}(C_W(t,\omega)) := \{x \in E : \ G(W(t,\omega))x \in \mathcal{D}(C(t))\},$$
$$C_W(t,\omega) := G^{-1}(W(t,\omega))C(t)G(W(t,\omega)),$$

where $W = (W_1, \dots, W_N)$. Note that the processes

$$G_W(t,\omega) := G(W(t,\omega)) \quad \text{and} \quad G_W^{-1}(t,\omega) := G(-W(t,\omega))$$

are adapted and pathwise strongly continuous.

In terms of the random operators $C_W(t)$ we introduce the following pathwise problem:

$$
\begin{aligned}
V'(t) &= C_W(t)V(t), \quad t \in [0,T], \\
V(0) &= u_0,
\end{aligned}
\tag{3.2}
$$

Notice that (3.2) is a special case of (3.1) with $A(t)$ replaced by $C_W(t)$ and with $B_n = 0$. In particular the notion of strong solution on $(0,T]$ and on $[0,T]$ apply.

Note that if V is a strong solution of (3.2) on $(0,T]$, then almost surely we have $G_W(t)V(t) \in \mathcal{D}(C(t)) = \mathcal{D}(A(t)) \subseteq \bigcap_{n=1}^N \mathcal{D}(B_n^2)$ for almost all $t \in [0,T]$.

The next theorem, which extends [11, Theorem 1] and [12, Theorem 1] to UMD Banach spaces, relates the problems (3.1) and (3.2).

Theorem 3.1. *Let E be a UMD Banach space and assume* (H1)–(H4) *and let $\varepsilon \in [0,T]$ be fixed. For a strongly measurable and adapted process $V : [0,T]\times\Omega \to E$ the following assertions are equivalent:*

(1) *$G_W V$ is a strong solution of* (3.1) *on $(0,T]$ (resp. on $[0,T]$);*
(2) *V is a strong solution of* (3.2) *on $(0,T]$ (resp. on $[0,T]$).*

Proof. First we claim that $\bigcap_{m,n=1}^N \mathcal{D}(B_n^* B_m^*)$ is norm-dense in E^*. Since E is reflexive it is sufficient to prove that $\bigcap_{m,n=1}^N \mathcal{D}(B_n^* B_m^*)$ is weak*-dense in E^*. Fix an $x \in E$, $x \neq 0$, and some $\lambda \in \bigcap_{n=1}^N \varrho(B_n)$, and put $y := \prod_{n=1}^N R(\lambda, B_n)^2 x$. Since by (H2) the resolvents $R(\lambda, B_n)$ commute we have $y \in \mathcal{D}\Big(\prod_{n=1}^N B_n^2\Big)$. Since $y \neq 0$ we can find $y^* \in E^*$ such that $\langle y, y^*\rangle \neq 0$. Then by (H2), the resolvents $R(\lambda, B_n^*)$ commute and $x^* := \prod_{n=1}^N R(\lambda, B_n^*)^2 y^* \in \bigcap_{m,n=1}^N \mathcal{D}(B_n^* B_m^*)$ and it is obvious that $\langle x, x^*\rangle \neq 0$. This proves the claim.

We will now turn to the proof of the equivalence of strong solutions on $(0,T]$. The equivalence of strong solutions on $[0,T]$ follows by taking $\varepsilon = 0$ in the proofs below.

(1) $\Rightarrow$ (2): Let $\varepsilon > 0$ be arbitrary. Since $U := G_W V$ is a strong solution of (3.1) on $(0,T]$, almost surely we have $G_W(t)V(t) \in \mathcal{D}(C(t))$ for almost all $t \in [\varepsilon, T]$. Moreover, for $n = 1, \dots, N$,

$$
\begin{aligned}
B_n^2 U(t) &= B_n^2 R(\lambda, A(t))(\lambda - A(t))U \\
&= B_n^2 R(\lambda, A(t))\lambda U(t) + B_n^2 R(\lambda, A(t))A(t)U(t).
\end{aligned}
$$

Therefore, (H4) implies that $B_n^2 G_W V = B_n^2 U$ is in $L^1(\varepsilon, T; E)$ almost surely. We conclude that $t \mapsto C(t)G_W(t)V(t)$ belongs to $L^1(\varepsilon, T; E)$ almost surely. Hence $t \mapsto C_W(t)V$ belongs to $L^1(\varepsilon, T; E)$ almost surely.

Let $x^* \in \bigcap_{m,n=1}^N \mathcal{D}(B_n^* B_m^*)$ be fixed. The function $f : \mathbb{R}^N \to E^*$ defined by $f(a) := G^{-1*}(a)x^*$ is twice continuously differentiable with

$$
\frac{\partial f}{\partial a_n}(a) = -G^{-1*}(a)B_n^* x^*, \qquad \frac{\partial^2 f}{\partial a_n^2}(a) = G^{-1*}(a)B_n^{*2} x^*.
$$

By the Itô formula Theorem 2.4 (applied to the Banach space E^* and the Hilbert space $H = \mathbb{R}^N$) it follows that the processes $G_W^{-1*} B_n^* x^*$ are stochastically integrable

with respect to W_n on $[\varepsilon, T]$ and that almost surely, for all $t \in [\varepsilon, T]$,

$$\begin{aligned}
&G_W^{-1*}(t)x^* - G_W^{-1*}(\varepsilon)x^* \\
&\qquad = -\sum_{n=1}^N \int_\varepsilon^t G_W^{-1*}(s)B_n^* x^* \, dW_n(s) + \frac{1}{2}\sum_{n=1}^N \int_\varepsilon^t G_W^{-1*}(s)B_n^{2*} x^* \, ds.
\end{aligned}$$

By (2.6) applied to U and $G_W^{-1*}x^*$ we obtain that almost surely, for all $t \in [\varepsilon, T]$,

$$\begin{aligned}
\langle V(t), x^*\rangle - \langle V(\varepsilon), x^*\rangle &= \langle U(t), G_W^{-1*}(t)x^*\rangle - \langle U(\varepsilon), G_W^{-1*}(\varepsilon)x^*\rangle \\
&= \int_\varepsilon^t \frac{1}{2}\sum_{n=1}^N \langle U(s), G_W^{-1*}(s)B_n^{*2}x^*\rangle + \langle A(s)U(s), G_W^{-1*}(s)x^*\rangle \, ds \\
&\qquad + \sum_{n=1}^N \int_\varepsilon^t -\langle U(s), G_W^{-1*}(s)B_n^* x^*\rangle + \langle B_n U(s), G_W^{-1*}(s)x^*\rangle \, dW_n(s) \\
&\qquad - \sum_{n=1}^N \int_\varepsilon^t \langle B_n U(s), G_W^{-1*}(s)B_n^* x^*\rangle \, ds \\
&= \int_\varepsilon^t \langle G_W^{-1}(s)C(s)U(s), x^*\rangle \, ds \\
&= \int_\varepsilon^t \langle C_W(s)V(s), x^*\rangle \, ds.
\end{aligned}$$

Since $C_W V$ has paths in $L^1(\varepsilon, T; E)$ almost surely, it follows that, almost surely, for all $t \in [\varepsilon, T]$,

$$\langle V(t), x^*\rangle - \langle V(\varepsilon), x^*\rangle = \Big\langle \int_\varepsilon^t C_W(s)V(s)\, ds, x^* \Big\rangle$$

By approximation this identity extends to arbitrary $x^* \in E^*$. By strong measurability, this shows that almost surely, for all $t \in [\varepsilon, T]$,

$$V(t) - V(\varepsilon) = \int_\varepsilon^t C_W(s)V(s)\, ds.$$

(2) $\Rightarrow$ (1): Put $U := G_W V$. Let $\varepsilon > 0$ be arbitrary. Since V is a strong solution of (3.2) on $(0, T]$, as before (H4) implies that almost surely we have $U(t) \in \mathcal{D}(A(t))$ for all $t \in [0, T]$ and $t \mapsto A(t)U(t)$ belongs to $L^1(\varepsilon, T; E)$.

Let $x^* \in \bigcap_{m,n=1}^N \mathcal{D}(B_n^* B_m^*)$ be fixed. Applying the Itô formula in the same way as before, the processes $G_W^* B_n^* x^*$ are stochastically integrable with respect to W_n on $[\varepsilon, T]$ and almost surely we have, for all $t \in [\varepsilon, T]$,

$$G_W^*(t)x^* - G_W^*(\varepsilon)x^* = \sum_{n=1}^N \int_\varepsilon^t G_W^*(s)B_n^* x^* \, dW_n(s) + \frac{1}{2}\sum_{n=1}^N \int_\varepsilon^t G_W^*(s)B_n^{*2}x^* \, ds.$$

By assumption we have $C_W V \in L^1(\varepsilon, T; E)$ almost surely. Hence we may apply (2.6) with V and $G_W^* x^*$. It follows that almost surely, for all $t \in [\varepsilon, T]$,

$$\begin{aligned}
&\langle U(t), x^*\rangle - \langle U(\varepsilon), x^*\rangle \\
&\quad = \langle V(t), G_W^*(t)x^*\rangle - \langle V(\varepsilon), G_W^*(t)x^*\rangle \\
&\quad = \int_\varepsilon^t \frac{1}{2}\sum_{n=1}^N \langle V(s), G_W^*(s)B_n^{*2}x^*\rangle + \langle G_W^{-1}(s)C(s)G_W(s)V(s), G_W^*(s)x^*\rangle \, ds \\
&\qquad\qquad + \sum_{n=1}^N \int_\varepsilon^t \langle V(s), G_W^*(s)B_n^* x^*\rangle \, dW_n(s) \\
&\quad = \int_\varepsilon^t \langle A(s)G_W(s)V(s), x^*\rangle \, ds + \sum_{n=1}^N \int_\varepsilon^t \langle B_n G_W(s)V(s), x^*\rangle \, dW_n(s) \\
&\quad = \int_\varepsilon^t \langle A(s)U(s), x^*\rangle \, ds + \sum_{n=1}^N \int_\varepsilon^t \langle B_n U(s), x^*\rangle \, dW_n(s).
\end{aligned}$$

Since $G_W^{-1}CU = C_W V \in L^1(\varepsilon, T; E)$ almost surely, we have $CU \in L^1(\varepsilon, T; E)$ almost surely, and therefore by (H4) we also have $AU \in L^1(\varepsilon, T; E)$ almost surely. Also, V has continuous paths almost surely, and therefore the same is true for $U = G_W V$. Thanks to the claim we are now in a position to apply Lemma 2.2 on the interval $[\varepsilon, T]$ (for the Hilbert space $H = \mathbb{R}^N$ and the process $\zeta = U - U(\varepsilon) - \int_\varepsilon^\cdot A(s)U(s)\, ds$). We obtain that the processes $B_n U$ are stochastically integrable with respect to W_n on $[\varepsilon, T]$ and that almost surely we have, for all $t \in [\varepsilon, T]$,

$$U(t) - U(\varepsilon) = \int_\varepsilon^t A(s)U(s)\, ds + \sum_{n=1}^N \int_\varepsilon^t B_n U(s)\, dW_n(s).$$

$\square$

4. The deterministic problem: Acquistapace-Terreni conditions

Consider the non-autonomous Cauchy problem:

$$\begin{aligned}
&\frac{du}{dt}(t) = \mathcal{C}(t)u(t) \quad t \in [0, T], \\
&u(0) = x,
\end{aligned} \tag{4.1}$$

where $\mathcal{C}(t) : \mathcal{D}(\mathcal{C}(t)) \subseteq E \to E$ are linear operators. We study this equation assuming the Acquistapace-Terreni conditions [3]:

(AT1) For all $t \in [0, T]$, $\mathcal{C}(t) : \mathcal{D}(\mathcal{C}(t)) \subseteq E \to E$ is a closed linear operator and there exists $\theta \in \left(\frac{\pi}{2}, \pi\right)$ such that for all $t \in [0, T]$ we have

$$\varrho(\mathcal{C}(t)) \supseteq \overline{S_\theta},$$

where $S_\theta = \{z \in \mathbb{C} \setminus \{0\} : \ |\arg z| < \theta\}$. Moreover there exists a constant $K \geq 0$ such that for all $t \in [0, T]$ we have

$$\|R(\lambda, \mathcal{C}(t))\| \leq \frac{K}{1 + |\lambda|}, \qquad \lambda \in S_\theta.$$

(AT2) There exist $k \geq 1$ and constants $L \geq 0$, $\alpha_1, \ldots, \alpha_k$, and $\beta_1, \ldots, \beta_k \in \mathbb{R}$ with $0 \leq \beta_i < \alpha_i \leq 2$ such that for all $t, s \in [0,T]$ we have

$$\|\mathcal{C}(t)R(\lambda, \mathcal{C}(t))[\mathcal{C}^{-1}(t) - \mathcal{C}^{-1}(s)]\| \leq L \sum_{i=1}^{k} |t-s|^{\alpha_i} |\lambda|^{\beta_i - 1}, \qquad \lambda \in S_\theta.$$

We may assume $\delta := \min\{\alpha_i - \beta_i\} \in (0,1)$.

We say that u is a *classical solution* of (4.1) if

(1) $u \in C([0,T]; E) \cap C^1((0,T], E)$;
(2) $u(t) \in \mathcal{D}(\mathcal{C}(t))$ for all $t \in (0,T]$;
(3) $u(0) = x$ and $u'(t) = \mathcal{C}(t)u(t)$ for all $t \in (0,T]$.

Assuming that $x \in \mathcal{D}(\mathcal{C}(0))$ we say that u is a *strict solution* of (4.1) if

(1) $u \in C^1([0,T]; E)$;
(2) $u(t) \in \mathcal{D}(\mathcal{C}(t))$ for all $t \in [0,T]$;
(3) $u(0) = x$ and $u'(t) = \mathcal{C}(t)u(t)$ for all $t \in [0,T]$.

As a special case of [3, Theorems 6.1, 6.3 and 6.5] and [1, Theorem 5.2] we have the following result. For a closed densely defined operator $(\mathscr{A}, \mathcal{D}(\mathscr{A}))$ on E we use the usual notation $D_{\mathscr{A}}(\theta, p) = (E, \mathcal{D}(\mathscr{A}))_{\theta,p}$ for the real interpolation spaces.

Theorem 4.1. *If the operators $\mathcal{C}(t) - \mu$ satisfy* (AT1) *and* (AT2) *for some $\mu \in \mathbb{R}$, then the following assertions hold:*

(1) *If $x \in \overline{\mathcal{D}(\mathcal{C}(0))}$, then there exists a unique classical solution u of* (4.1).
(2) *If $x \in \mathcal{D}_{\mathcal{C}(0)}(1-\sigma, \infty)$ with $\sigma \in (0,1)$, then there exists a unique classical solution u of* (4.1). *Moreover $\mathcal{C}u \in L^p(0,T;E)$ for all $1 \leq p < \sigma^{-1}$.*
(3) *If $x \in \mathcal{D}(\mathcal{C}(0))$, then there exists a unique strict solution u of* (4.1).

Assuming Hypothesis (H2), we study the problem

$$\begin{aligned} \frac{du}{dt}(t) &= \mathcal{C}_h(t)u(t) \quad t \in [0,T], \\ u(0) &= x, \end{aligned} \tag{4.2}$$

Here $\mathcal{C}_h(t) = G^{-1}(h(t))\mathcal{C}(t)G(h(t))$, with $\mathcal{D}(\mathcal{C}_h(t)) = \{x \in E : G(h(t))x \in \mathcal{D}(\mathcal{C}(t))\}$, G is as in Section 3, and $h : [0,T] \to \mathbb{R}^N$ is a measurable function. Notice that (3.2) may be seen as the special case of (4.2), where $\mathcal{C} = C$ and $h = W$.

The following condition is introduced in [13, Theorem 6.30] (see also [11, Proposition 1]) in the time independent case. Let $(\mathcal{C}(t))_{t\in[0,T]}$ be densely defined and such that $0 \in \varrho(\mathcal{C}(t))$ for all $t \in [0,T]$. Assuming Hypothesis (H2) we consider the following Hypothesis (K) (which may be weakened somewhat, cf. [2, Remark 1.2]).

(K) We have $0 \in \varrho(\mathcal{C}(t))$ for all $t \in [0,T]$ and there exist uniformly bounded functions $K_n : [0,T] \to \mathcal{L}(E)$ such that for all $t \in [0,T]$, all $n = 1, \ldots, N$, and all $x \in \mathcal{D}(B_n)$ we have $B_n\mathcal{C}^{-1}(t)x \in \mathcal{D}(\mathcal{C}(t))$ and

$$\mathcal{C}(t)B_n\mathcal{C}^{-1}(t)x = B_n x + K_n(t)x.$$

The latter may be rewritten as the commutator condition:

$$[\mathcal{C}(t), B_n]\mathcal{C}^{-1}(t)x = K_n(t)x.$$

In many cases it is enough to consider only $x \in \mathcal{D}(\mathcal{C}(t))$ instead of $x \in \mathcal{D}(B_n)$ (cf. [2, Proposition A.1]).

Assume that (AT1) and (AT2) hold for the operators $\mathcal{C}(t)$. If (K) holds for the operators $\mathcal{C}(t)$, then the uniform boundedness of $t \mapsto R(\lambda, \mathcal{C}(t))$ can be used to check that for all $\lambda > 0$, (K) holds for the operators $\mathcal{C}(t) - \lambda$ for all $\lambda > 0$.

The following lemma lists some consequences of Hypothesis (K).

Lemma 4.2. *Let $(\mathcal{C}(t))_{t\in[0,T]}$ be closed densely defined operators such that $0 \in \varrho(A(t))$ for all $t \in [0,T]$. Assume Hypotheses* (H2) *and* (K)*.*

(1) *For all $n = 1, \ldots, N$, $s \in \mathbb{R}$ and $t \in [0,T]$, $G_n(s)$ leaves $\mathcal{D}(\mathcal{C}(t))$ invariant and*
$$\mathcal{C}(t)G_n(s)\mathcal{C}^{-1}(t) = e^{s(B_n + K_n(t))}.$$
(2) *For all $R \geq 0$ there is a constant $M_R \geq 0$ such that for all $n = 1, \ldots, N$, $|s| \leq R$ and $t \in [0,T]$ we have*
$$\|\mathcal{C}(t)G_n(s)\mathcal{C}^{-1}(t) - G_n(s)\| \leq M_R|s|.$$

Proof. The first assertion follows from the proof of [11, Proposition 1] and the second from a standard perturbation result, cf. [16, Corollary III.1.11]. ☐

We can now formulate a result that related the problems (4.1) and (4.2).

Proposition 4.3. *Let $(\mathcal{C}(t))_{t\in[0,T]}$ be closed densely defined operators such that $0 \in \varrho(\mathcal{C}(t))$ for all $t \in [0,T]$. Assume Hypotheses* (H2) *and* (K)*. Let $h : [0,T] \to \mathbb{R}^N$ be Hölder continuous with parameter $\alpha \in (0,1]$ and define the similar operators $(\mathcal{C}_h(t))_{t\in[0,T]}$ as*
$$\mathcal{C}_h(t) = G^{-1}(h(t))\mathcal{C}(t)G(h(t)) \quad \text{with } D(\mathcal{C}_h(t)) = \{x \in E : G(h(t))x \in D(\mathcal{C}(t))\}.$$
If the operators $\mathcal{C}(t)$ satisfy (AT1) *and* (AT2) *with $[(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)]$, then the operators $\mathcal{C}_h(t)$ satisfy* (AT1) *and* (AT2) *with $[(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k), (\alpha, 0)]$.*

Proof. We denote $G_h(t) = G(h(t))$. For all $t \in [0,T]$ and $\lambda \in \varrho(\mathcal{C}(t))$ we clearly have $\lambda \in \varrho(\mathcal{C}_h(t))$ and $R(\lambda, \mathcal{C}_h(t)) = G_h^{-1}(t)R(\lambda, \mathcal{C}(t))G_h(t)$. It follows from the assumptions on h that for all $t \in [0,T]$,
$$\|R(\lambda, \mathcal{C}_h(t))\| \leq M^2\|R(\lambda, \mathcal{C}(t))\|,$$
where $M = \sup_{t\in[0,T]} \|G(h(t))\| \vee \|G(-h(t))\|$. Hence each $\mathcal{C}_h(t)$ is a sectorial operator with the same sector as $\mathcal{C}(t)$. Thus the operators $\mathcal{C}_h(t)$ satisfy (AT1).

Next we check (AT2). For all $t, s \in [0,T]$ we have
$$\begin{aligned}
&\|\mathcal{C}_h(t)R(\lambda, \mathcal{C}_h(t))[\mathcal{C}_h^{-1}(t) - \mathcal{C}_h^{-1}(s)]\| \\
&\quad = \|G_h^{-1}(t)\mathcal{C}(t)R(\lambda, \mathcal{C}(t))[\mathcal{C}^{-1}(t)G_h(t) - G_h(t)G_h^{-1}(s)\mathcal{C}^{-1}(s)G_h(s)]\| \\
&\quad \leq M\|\mathcal{C}(t)R(\lambda, \mathcal{C}(t))[\mathcal{C}^{-1}(t)G_h(t) - \mathcal{C}^{-1}(s)G_h(t)]\| \\
&\qquad + M\|\mathcal{C}(t)R(\lambda, \mathcal{C}(t))[\mathcal{C}^{-1}(s)G_h(t) - G_h(t)G_h^{-1}(s)\mathcal{C}^{-1}(s)G_h(s)]\|.
\end{aligned}$$
We estimate the two terms on the right-hand side separately. Since $(\mathcal{C}(t))_{t\in[0,T]}$ satisfies (AT2), it follows for the first term that
$$\begin{aligned}
&\|\mathcal{C}(t)R(\lambda, \mathcal{C}(t))[\mathcal{C}^{-1}(t)G_h(t) - \mathcal{C}^{-1}(s)G_h(t)]\| \\
&\qquad \leq M\|\mathcal{C}(t)R(\lambda, \mathcal{C}(t))[\mathcal{C}^{-1}(t) - \mathcal{C}^{-1}(s)]\| \\
&\qquad \leq ML\sum_{i=1}^{k} |t-s|^{\alpha_i}|\lambda|^{\beta_i - 1}.
\end{aligned} \tag{4.3}$$

For the second term we have

$$\begin{aligned}
&\|\mathcal{C}(t)R(\lambda,\mathcal{C}(t))[\mathcal{C}^{-1}(s)G_h(t) - G_h(t)G_h^{-1}(s)\mathcal{C}^{-1}(s)G_h(s)]\| \\
&\quad\le M\|\mathcal{C}(t)R(\lambda,\mathcal{C}(t))\mathcal{C}^{-1}(s)[G_h(t)G_h^{-1}(s) - \mathcal{C}(s)G_h(t)G_h^{-1}(s)\mathcal{C}^{-1}(s)]\| \\
&\quad= M\|\mathcal{C}(t)R(\lambda,\mathcal{C}(t))\mathcal{C}^{-1}(s)[G(h(t)-h(s)) - \mathcal{C}(s)G_h(t)G_h^{-1}(s)\mathcal{C}^{-1}(s)]\|.
\end{aligned} \tag{4.4}$$

By an induction argument and Lemma 4.2 as in the proof of [13, Theorem 6.30], the Hölder continuity of h implies that for all $t, s \in [0,T]$,

$$\|G(h(t)-h(s)) - \mathcal{C}(s)G_h(t)G_h^{-1}(s)\mathcal{C}^{-1}(s)\| \le M_\alpha N|t-s|^\alpha. \tag{4.5}$$

On the other hand it follows from (AT1) and (AT2) that

$$\begin{aligned}
&\|\mathcal{C}(t)R(\lambda,\mathcal{C}(t))\mathcal{C}^{-1}(s)\| \\
&\quad\le \|\mathcal{C}(t)R(\lambda,\mathcal{C}(t))[\mathcal{C}^{-1}(s)-\mathcal{C}^{-1}(t)]\| + \|\mathcal{C}(t)R(\lambda,\mathcal{C}(t))\mathcal{C}^{-1}(t)\| \\
&\quad\le L\sum_{i=1}^k |t-s|^{\alpha_i}|\lambda|^{\beta_i-1} + K|\lambda|^{-1}.
\end{aligned} \tag{4.6}$$

Combining (4.4), (4.5) and (4.6) gives

$$\begin{aligned}
&\|\mathcal{C}(t)R(\lambda,\mathcal{C}(t))[\mathcal{C}^{-1}(s)G_h(t) - G_h(t)G_h^{-1}(s)\mathcal{C}^{-1}(s)G_h(s)\| \\
&\qquad\le MLM_\alpha N\sum_{i=1}^k |t-s|^{\alpha_i+\alpha}|\lambda|^{\beta_i-1} + MKM_\alpha N|t-s|^\alpha|\lambda|^{-1}
\end{aligned} \tag{4.7}$$

We conclude from (4.3), (4.7), and the trivial estimate $|t-s|^{\alpha_i+\alpha} \le C_T|t-s|^{\alpha_i}$ that

$$\big\|\mathcal{C}_h(t)R(\lambda,\mathcal{C}_h(t))\big[\mathcal{C}_h^{-1}(s)-\mathcal{C}_h^{-1}(t)\big]\big\| \le \tilde{L}\sum_{i=1}^{k+1}|t-s|^{\alpha_i}|\lambda|^{\beta_i-1}$$

for a certain constant $\tilde{L}$ and $\alpha_{k+1}=\alpha$, $\beta_{k+1}=0$. ☐

The main abstract result of this paper reads as follows.

Theorem 4.4. *Let E be a UMD Banach space and assume that Hypotheses* (H1)*,* (H2)*,* (H3)*, and* (H4) *are fulfilled and that* (AT1)*,* (AT2)*, and* (K) *are satisfied for $C(t)-\mu$ for some $\mu \in \mathbb{R}$.*

1. *The problem* (3.1) *admits a unique strong solution U on $(0,T]$ for which $AU \in C((0,T];E)$ almost surely.*
2. *If $u_0 \in \mathcal{D}_{A(0)}(1-\sigma,\infty)$ almost surely, then the problem* (3.1) *admits a unique strong solution U on $[0,T]$ for which $AU \in C((0,T];E)$ almost surely. Moreover $AU \in L^p(0,T;E)$ for all $1 \le p < \sigma^{-1}$.*
3. *If $u_0 \in \mathcal{D}(A(0))$ almost surely, the problem* (3.1) *admits a unique strong solution U on $[0,T]$ for which $AU \in C([0,T];E)$ almost surely.*

Proof. If U_μ is a solution of (3.1) with $A(t)$ replaced by $A(t)-\mu$, then it is easy to see that $t \mapsto e^{\mu t}U_\mu(t)$ is a solution of (3.1). It follows from this that without loss of generality we may assume that $\mu = 0$ in the assumptions above.

(1): By the standing assumption made in Section 3, the initial value u_0 is an $\mathcal{F}_0$-measurable random variable. By Proposition 4.3 and the Hölder continuity of Brownian motion, the operators $C_W(t)$ satisfy (AT1) and (AT2). Hence by

Theorem 4.1, almost surely the problem (3.2) admits a unique classical solution V. To see that V is adapted we argue as follows.

Let $(P_W(t,s))_{0\le s\le t\le T}$ be the evolution system generated by $(C_W(t))_{0\le t\le T}$, which exists by virtue of (AT1), (AT2), and the results of [1, 3]. Then $V(t) = P_W(t,0)u_0$. Thus we need to check that for each $t \in [0,T]$ the random variable $P_W(t,0)u_0$ is strongly $\mathcal{F}_t$-measurable. Since $u_0 : \Omega \to E$ is strongly $\mathcal{F}_0$-measurable we can approximate u_0 almost surely with $\mathcal{F}_0$-measurable simple functions. In this way the problem reduces to showing that $P_W(t,0)x$ is $\mathcal{F}_t$-measurable for all $x \in E$. One easily checks that the Yosida approximations $(C_W^{(n)}(s))_{s\in[0,t]}$ of $(C_W(s))_{s\in[0,t]}$ are strongly $\mathcal{F}_t$-measurable in the strong operator topology. Moreover, in view of (AT1) and (AT2), $C_W^{(n)}$ is almost surely (Hölder) continuous in the uniform operator topology. Therefore by the construction of the evolution family $P_W^{(n)}(u,s)_{0\le s\le u\le t}$ (for instance via the Banach fixed point theorem (cf. [27])) we obtain that $P_W^{(n)}(t,0)x$ is strongly $\mathcal{F}_t$-measurable. By [7, Proposition 4.4], $P_W(t,0)x = \lim_{n\to\infty} P_W^{(n)}(t,0)x$. This implies that $P_W(t,0)x$ is strongly $\mathcal{F}_t$-measurable.

Since V has continuous paths almost surely, it follows that V is strongly measurable. Since continuous functions are integrable, the solution V is a strong solution on $(0,T]$. Hence by Theorem 3.1, $U = G_W V$ is a strong solution of (3.1) on $(0,T]$. The pathwise regularity properties of V carry over to U, thanks to (H4). The pathwise uniqueness of V implies the uniqueness of U again via Theorem 3.1 and (H4).

(2): If $u_0 \in \mathcal{D}_{A(0)}(1-\sigma,\infty)$ almost surely, then it follows from $AV \in L^p(0,T;E)$ that V is a strong solution of (3.2) on $[0,T]$. Therefore, Theorem 3.1 implies that U is a strong solution of (3.1) on $[0,T]$. The pathwise regularity properties of V carry over to U as before.

(3): If $u_0 \in \mathcal{D}(A(0))$ almost surely, then V is a strong solution of (3.2) on $[0,T]$, and from Theorem 3.1 we see that U is a strong solution of (3.1) on $[0,T]$. The pathwise regularity properties of V carry over to U as before. □

Remark 4.5. If $(A(t)-\mu_0)_{t\in[0,T]}$ satisfies (AT1) and (AT2) for a certain $\mu_0 \in \mathbb{R}$, then under certain conditions the perturbation result in [14, Lemma 4.1] may be used to obtain that $(C(t)-\mu)_{t\in[0,T]}$ satisfies (AT1) and (AT2) as well for μ large enough. In particular, this is the case if the $(B_n)_{n=1}^N$ are assumed to be bounded.

Remark 4.6. Assume E is reflexive (e.g. E is a UMD space). If the B_n are bounded and commuting and the closed operators $A(t)-\mu_0$ and $C(t)-\mu_0$ satisfy (AT1), (AT2) for all $\mu_0 \in \mathbb{R}$ large enough, then (H1) - (H4) are fulfilled. It is trivial that (H2) and (H3) are satisfied. For (H1) one may use Kato's result (cf. [34, Section VIII.4]) to check the denseness of the domains. For (H4) notice that for $\lambda > \mu_0$ (AT1) and (AT2) imply that $t \mapsto R(\lambda, A(t))$ and $t \mapsto R(\lambda, C(t))$ are continuous (cf. [31, Lemma 6.7]). Since B_n are assumed to be bounded this clearly implies (H4).

Remark 4.7. Assume the operators $B_1, B_2, \ldots, B_N$ are bounded and commuting. Then each e^{tB_n} is continuously differentiable, so $G(W)$ is Hölder continuous with exponent $\mu \in (0, \frac{1}{2})$. As a consequence, time regularity of the solution V of (3.2) translates in time regularity of the solution $U = G(W)V$ of (3.1). We will illustrate this in two ways below.

As in [29, p. 5] it can be seen that if almost surely $u_0 \in D((w - A(0)^\alpha)$ for some $\alpha \in (0,1]$, then almost surely V is Hölder continuous with parameter α. We conclude that under the condition that almost surely, $u_0 \in D((w-A(0))^\alpha$ for some $\alpha \in (0,\frac{1}{2})$, U is Hölder continuous with parameter α.

Assume $u_0 \in D(A(0))$ and $A(0)u_0 \in D_A(\alpha,\infty)$ almost surely for some $\alpha \in (0,\delta]$. Then we deduce from [3, Section 6] that almost surely, $C_W V$ has paths in $C^\alpha([0,T];E)$. If $\alpha < \frac{1}{2}$, then we readily obtain, almost surely, AU has paths in $C^\alpha([0,T];E)$.

We conclude this section with an example. An non-stochastic version of the example has been studied in [1, 29, 33].

Example 4.8. We consider the problem

$$
\begin{aligned}
D_t u(t,x) &= A(t,x,D)U(t,x) + \sum_{n=1}^{N} B_n(x)U(t,x)D_tW_n(t), \quad t\in[0,T], x\in S \\
V(t,x,D)U &= 0, \quad t\in[0,T], x\in\partial S, \\
U(0,x) &= u_0(x), \quad x\in S
\end{aligned}
\tag{4.8}
$$

Here

$$A(t,x,D) = \sum_{i,j=1}^{d} a_{ij}(t,x)D_iD_j + \sum_{i=1}^{d} q_i(t,x)D_i + r(t,x), \quad B_n(x) = b_n(x),$$

and

$$V(t,x) = \sum_{i=1}^{d} v_i(t,x,D)D_i + v_0(t,x).$$

The set $S \subseteq \mathbb{R}^d$ is a bounded domain with boundary of class C^2 being locally on one side of S and outer unit normal vector $n(x)$. We assume that ∂S consists of two closed (possibly empty) disjoint subsets Γ_0 and Γ_1. Moreover the coefficients are real and $a_{ij}, q_i, r \in C^\alpha([0,T], C(\overline{S}))$, where $\alpha \in (\frac{1}{2},1)$ if $\Gamma_1 \neq \emptyset$ and $\alpha\in(0,1)$ if $\Gamma_1 = \emptyset$ and the matrix $(a_{ij}(\cdot,x))_{i,j}$ is symmetric and strictly positive definite uniformly in time, i.e. there exists an $\nu > 0$ such that for all $t\in[0,T]$ we have

$$\sum_{i,j=1}^{d} a_{ij}(t,x)\lambda_i\lambda_j \geq \nu \sum_{i=1}^{d} \lambda_i^2, \quad x\in S, \lambda\in\mathbb{R}^d.$$

The boundary coefficients are assumed to be real and $v_i, v_0 \in C^\alpha([0,T], C^1(\partial S))$, $v_0 = 1$ and $v_i = 0$ on Γ_0 and there is a constant $\delta > 0$ such that for all $x\in\Gamma_1$ and $t\in[0,T]$ we have $\sum_{i=1}^d v_i(t,x)n_i(x) \geq \delta$. Finally we assume that $b_n \in C^2(\overline{S})$ and

$$\sum_{i=1}^{d} v_i(t,x)D_ib_n(x) = 0, \quad t\in[0,T], x\in\partial S. \tag{4.9}$$

Under these assumptions, for all $p\in(1,\infty)$ and $u_0 \in L^0(\Omega;\mathcal{F}_0;L^p(S))$ there exists a unique strong solution U of (4.8) on $(0,T]$ for which $AU \in C((0,T];L^p(S))$ almost surely.

If $u_0 \in B^{2(1-\sigma)}_{p,\infty,\{V\}}(S)$ almost surely for some $\sigma\in(0,1)$ (see [32, Section 4.3.3] for the definition of this space) then there exists a unique strong solution U of (4.8)

on $[0,T]$ for which almost surely $AU \in C((0,T];L^p(S))$ and $AU \in L^q(0,T;L^p(S))$ for all $1 \leq q < \sigma^{-1}$.

Furthermore, if almost surely we have $u_0 \in W^{2,p}(S)$ and $V(0,x)u_0 = 0 \;\; x \in \partial S$, then there exists a unique strong solution U of (4.8) on $[0,T]$ for which $AU \in C([0,T];L^p(S))$ almost surely.

Finally, we notice that Remark 4.7 can be used to obtain time regularity of U and AU under conditions on u_0.

Proof. We check the conditions in Theorem 4.4, with $\mathcal{D}(A(t)) = \{f \in W^{2,p}(S) : V(t,\cdot)f = 0 \text{ on } \partial S\}$. If $\sigma \neq \frac{1}{2}(1-\frac{1}{p})$ (which can be assumed without loss of generality by replacing σ by a slighly larger value) $D_{A(0)}(1-\sigma,\infty) = B^{2(1-\sigma)}_{p,\infty,\{V\}}(S)$, cf. [32, Theorem 4.3.3].

It is shown in [29] that for $\lambda_0 \in \mathbb{R}$ large enough, (AT1) and (AT2) hold for $A(t) - \lambda_0$ and $C(t) - \lambda_0$, with coefficients α and $\beta = \frac{1}{2}$ in case $\Gamma_1 \neq \emptyset$ and $\beta = 0$ in case $\Gamma_1 = \emptyset$. Since the operators B_n are bounded, Remark 4.6 applies and we conclude that (H1)–(H4) hold.

Let $\lambda > \lambda_0$ be fixed. The only thing that is left to be checked is condition (K) for the operators $C(t) - \lambda$. It follows from (4.9) that for all $x \in E$, $B_n R(\lambda, C(t))x \in \mathcal{D}(C(t))$. For $n = 1,2,\dots,N$ and $t \in [0,T]$ define

$$K_n(t) = (C(t) - \lambda)B_n(C(t) - \lambda)^{-1} - B_n.$$

One can check that $K_n(t) = [C(t), B_n]R(\lambda, C(t))$, where $[C(t), B_n]$ is the commutator of $C(t)$ and B_n. Since $[C(t), B_n]$ is a first order operator, each $K_n(t)$ is a bounded operator. To prove their uniform boundedness in t, we note that from the assumptions on the coefficients it follows that there are constants $C_1, C_2 > 0$ such that for all $t \in [0,T]$ and $j = 1,\dots,d$,

$$\|R(\lambda, C(t))\| \leq C_1 \text{ and } \|D_j R(\lambda, C(t))\| \leq C_2.$$

Indeed, the first estimate is obviously true, and the second one follows from the Agmon-Douglis-Nirenberg estimates (see [4]). □

5. The deterministic problem: Tanabe conditions

In the theory for operators $\mathcal{C}(t)$ with time-independent domains $\mathcal{D}(\mathcal{C}(t)) =: \mathcal{D}(\mathcal{C}(0))$ (cf. [30, Section 5.2], see also [5, 22, 27]), condition (AT2) is often replaced by the following stronger condition, usually called the Tanabe condition,

(T2) There are constants $L \geq 0$ and $\mu \in (0,1]$ such that for all $t,s \in [0,T]$ we have
$$\|\mathcal{C}(t)\mathcal{C}^{-1}(0) - \mathcal{C}(s)\mathcal{C}^{-1}(0)\| \leq L|t-s|^{\mu}.$$

It is shown in [30] that condition (T2) implies that there is a constant $\tilde{L} \geq 0$, such that for all $t,s,r \in [0,T]$ we have

$$\|\mathcal{C}(t)\mathcal{C}^{-1}(r) - \mathcal{C}(s)\mathcal{C}^{-1}(r)\| \leq \tilde{L}|t-s|^{\mu}. \tag{5.1}$$

In particular the family $\{\mathcal{C}(s)\mathcal{C}^{-1}(t) : s,t \in [0,T]\}$ is uniformly bounded.

It is clear that under (H1) and (H3), the operators $A(t)$ satisfy (T2) if and only if the operators $C(t)$ satisfy (T2).

Lemma 5.1. *Assume* (H1), (H3) *and that* $\mathcal{D}(A(t)) = \mathcal{D}(A(0))$. *If* $(A(t))_{t\in[0,T]}$ *and* $(C(t))_{t\in[0,T]}$ *satisfy* (AT1) *and* (T2), *then* (H4) *holds.*

Proof. Since $\mathcal{D}(A(0)) \subseteq \mathcal{D}(B_n^2)$ and $0 \in \varrho(A(0))$, there is a constant C_n such that $\|B_n^2 x\| \leq C_n \|A(0)x\|$ for all $x \in D(A(0))$. It follows from the uniform boundedness of $\{A(0)A^{-1}(t) : t \in [0,T]\}$ and (5.1) that for all $t, s \in [0,T]$ we have

$$\begin{aligned}
\|B_n^2 A^{-1}(t) - B_n^2 A^{-1}(s)\| &\leq C_n \|A(0)A^{-1}(t) - A(0)A^{-1}(s)\| \\
&\leq C_n C \|(A(t)A^{-1}(t) - A(t)A^{-1}(s))\| \\
&\leq C_n C \|(A(s)A^{-1}(s) - A(t)A^{-1}(s))\| \leq C_n C \tilde{L} |t-s|^{\mu}.
\end{aligned}$$

This shows that $t \mapsto B_n^2 A^{-1}(t)$ is Hölder continuous. In the same way one can show that $t \mapsto B_n^2 C^{-1}(t)$ is Hölder continuous. We conclude that (H4) holds. □

It is easy to see that the statement in Proposition 4.3 holds as well with (AT2) replaced by (T2) (in the assumption and the assertion). Thus in the case where the domains $\mathcal{D}(A(t))$ are constant, the more difficult Acquistapace-Terreni theory is not needed.

If the operators $B_1, \ldots, B_N$ are bounded we have the following consequence of Theorem 4.4. Note that the assumptions are made on the operators $A(t)$ rather than on $C(t)$.

Proposition 5.2. *Let E be a UMD space and $\mathcal{D}(A(t)) = \mathcal{D}(A(0))$. Assume that the operators $A(t) - \lambda$ satisfy* (AT1) *and* (T2) *for all $\lambda \in \mathbb{R}$ large enough, and let $B_1, \ldots, B_N \in \mathcal{L}(E)$ be bounded commuting operators which leave $\mathcal{D}(A(0))$ invariant. Consider the problem*

$$\begin{aligned}
dU(t) &= A(t)U(t)\,dt + \sum_{n=1}^{N} B_n U(t) dW_n(t), \quad t \in [0,T], \\
U(0) &= u_0.
\end{aligned} \tag{5.2}$$

(1) *If $u_0 \in E$ almost surely, the problem* (5.2) *admits a unique strong solution $U \in C([0,T];E)$ on $(0,T]$ for which $AU \in C((0,T];E)$.*
(2) *If $u_0 \in \mathcal{D}_{A(0)}(1-\sigma,\infty)$ almost surely, then the problem* (3.1) *admits a unique strong solution $U \in C([0,T];E)$ on $[0,T]$ with $AU \in C((0,T];E)$. Moreover $AU \in L^p(0,T;E)$ for all $1 \leq p < \sigma^{-1}$.*
(3) *If $u_0 \in \mathcal{D}(A)$ almost surely, the problem* (3.1) *admits a unique strong solution $U \in C([0,T];E)$ on $[0,T]$ for which $AU \in C([0,T];E)$.*

Proof. We check the conditions of Theorem 4.4. It follows from Remark 4.6 that (H1), (H2) and (H3) are satisfied. Lemma 5.1 implies that (H4) is satisfied.

By the bounded perturbation theorem, for $\lambda \in \mathbb{R}$ large enough the operators $C(t) - \lambda = A(t) - \frac{1}{2}\sum_{n=1}^{N} B_n^2 - \lambda$ satisfy (AT1). Hence for λ large enough, condition (T2) for the operators $C(t) - \lambda$ follows from (T2) for the operators $A(t) - \lambda$.

Finally to check (K), by the assumption on the operators B_n we have $\mathcal{D}(A(0)) = \mathcal{D}(C(0))$, and by the closed graph theorem we have $\|B_n x\|_{\mathcal{D}(C(0))} \leq c_n \|x\|_{\mathcal{D}(C(0))}$ for some constant c_n. This implies that $\|C(0)B_n x\| \leq c_n \|C(0)x\|$. We check that the operators $K_n(t) = C(t)B_n C^{-1}(t) - B_n$ are uniformly bounded. By the remark following (5.1), the family $\{C(0)C^{-1}(t) : t \in [0,T]\}$ is uniformly bounded, say by some constant k, and therefore

$$\begin{aligned}
\|C(t)B_n C^{-1}(t)\| &\leq \|C(t)C^{-1}(0)C(0)B_n C^{-1}(0)C(0)C^{-1}(t)\| \\
&\leq k^2 \|C(0)B_n C^{-1}(0)\| \leq c_n.
\end{aligned}$$

□

Next we return to the problem (1.1) discussed at the beginning of the paper.

Example 5.3. We consider the problem

$$
\begin{aligned}
D_t u(t,x) &= A(t,x,D)U(t,x) + B(x,D)D_t W(t), \quad t\in[0,T], x\in\mathbb{R}^d \\
U(0,x) &= u_0(x), \quad x\in\mathbb{R}^d
\end{aligned}
\tag{5.3}
$$

Here

$$
\begin{aligned}
A(t,x,D) &= \sum_{i,j=1}^d a_{ij}(t,x)D_iD_j + \sum_{i=1}^d q_i(t,x)D_i + r(t,x),\\
B(x,D) &= \sum_{i=1}^d b_i(x)D_i + c(x).
\end{aligned}
$$

All coefficients are real-valued and we take a_{ij}, q_i, r uniformly bounded in time with values in $C^1_b(R^d))$. The coefficients a_{ij}, q_i and r are μ-Hölder continuous in time for some $\mu\in(0,1]$, uniformly in $\mathbb{R}^d$. Furthermore we assume that the matrices $(a_{ij}(t,x))_{i,j}$ are symmetric, and there exists a constant $\nu>0$ such that for all $t\in[0,T]$

$$
\sum_{i,j=1}^d \Big(a_{ij}(t,x) - \frac12 b_i(x)b_j(x)\Big)\lambda_i\lambda_j \geq \nu\sum_{i=1}^d \lambda_i^2, \quad x\in\mathbb{R}^d, \lambda\in\mathbb{R}^d.
$$

Finally, we assume that $b_i, c\in C^2_b(\mathbb{R}^d)$. Under these assumptions it follows from Theorem 4.4 that for all $p\in(1,\infty)$ and $u_0\in L^0(\Omega,\mathcal{F}_0;L^p(\mathbb{R}^d))$, there exists a unique strong solution U of (5.3) on $(0,T]$ with paths in $C([0,T];L^p(\mathbb{R}^d))\cap C((0,T];W^{2,p}(\mathbb{R}^d))$. If moreover $u_0\in B^{2(1-\sigma)}_{p,\infty}(\mathbb{R}^d)$ almost surely, then there exists a unique strong solution U of (4.8) on $[0,T]$ for which $U\in C((0,T];W^{2,p}(\mathbb{R}^d))$ almost surely and $AU\in L^q(0,T;L^p(\mathbb{R}^d))$ for all $1\leq q<\sigma^{-1}$. If $u_0\in W^{2,p}(\mathbb{R}^d)$ almost surely, then there exists a unique strong solution U of (5.3) on $[0,T]$ with paths in $C^\alpha([0,T];L^p(\mathbb{R}^d))\cap C([0,T];W^{2,p}(\mathbb{R}^d))$ for all $\alpha\in(0,\frac12)$.

In [8], for $A(t)\equiv A$ a strong solution on $[0,T]$ with paths in $L^2(0,T;W^{2,p}(\mathbb{R}^d))$ almost surely is obtained for initial data satisfying $u_0\in B^1_{p,2}(\mathbb{R}^d)$ almost surely. In [19] it is assumed that $u_0\in H_p^{2-\frac{2}{p}}(\mathbb{R}^d)$ and a solution is obtained with paths in $L^p(0,T;W^{2,p}(\mathbb{R}^d))$ almost surely.

Proof. Let $E=L^p(\mathbb{R}^d)$, where $p\in(1,\infty)$. Let $\mathcal{D}(A(t))=W^{2,p}(\mathbb{R}^d)$ and $A(t)f)=A(t,\cdot,D)f$ for all $t\in[0,T]$. Let $\mathcal{D}(B_0)=W^{1,p}(\mathbb{R}^d)$ and $B_0f=B(\cdot,D)f$, and let $(B,D(B))$ be the closure of $(B_0,D(B_0))$. Note that by real interpolation we have $B^{2(1-\sigma)}_{p,\infty}(\mathbb{R}^d)=D_A(1-\sigma,\infty)$, see [32].

We check the conditions of Theorem 4.4. We begin with the Hypotheses (H1)-(H3). That (H1) holds is clear, and (H2) follows as in [6, Example C.III.4.12]. Finally (H3) follows from $\mathcal{D}(A(t))\subseteq\mathcal{D}(B^2)$.

The operators $A(t)-\lambda$ and $C(t)-\lambda$ satisfy condition (AT1) for all $\lambda\in\mathbb{R}$ large enough (cf. [22, Section 3.1]). Furthermore it can be checked that $A(t)-\lambda$ and $C(t)-\lambda$ satisfy (T2). Now Condition (H4) follows from (5.1).

To check (K) for the operators $C(t)-\lambda$, put $K(t)=[C(t),B]R(\lambda,C(t))$. Since the third order derivatives in the commutator $[C(t),B]$ cancel and $a_{ij}(t), q_i(t), r(t)\in$

$C^1_b(R^d)$ and $b_i, c \in C^2_b(\mathbb{R}^d)$, the operators $K(t)$ are bounded for each $t \in [0,T]$. Moreover,

$$K(t) = [C(t), B]R(\lambda, C(t)) = [C(t)-\lambda, B]R(\lambda, C(t)) = (C(t)-\lambda)B(C(t)-\lambda)^{-1}+B$$

on $W^{1,p}(\mathbb{R}^d)$, and this identity extends to $\mathcal{D}(B)$ (see [2, Proposition A.1]). To check that K is uniformly bounded, note that by the uniform boundedness of the family $(\lambda - C(0))R(\lambda, C(t))$ it suffices to check that there is a constant C such that for all $t \in [0,T]$ and $f \in W^{2,p}(\mathbb{R}^d)$,

$$\|[C(t), B]f\| \leq C\|f\|_{W^{2,p}(\mathbb{R}^d)}.$$

But this follows from the assumptions $a_{ij}, q_i, r \in \mathcal{L}^\infty([0,T]; C^1_b(R^d))$ and $b_i, c \in C^2_b(\mathbb{R}^d)$.

Finally, we show that if $u_0 \in W^{2,p}(\mathbb{R}^d)$ almost surely, then U has paths in $C^\alpha([0,T]; L^p(\mathbb{R}^d))$ for all $\alpha \in (0, \frac{1}{2})$. One can check that for all $x \in D(A(0))$, $G(t)x$ is continuously differentiable and there are constants C_1, C_2 such that for all $x \in D(A(0))$ and $s,t \in [0,T]$,

$$\|G(t)x - G(s)x\| \leq C_1|t-s|\|x\|_{\mathcal{D}(A(0))} \leq C_2|t-s|\|x\|_{\mathcal{D}(C_W(0))}.$$

On the other hand it follows from Theorem 4.1 that (3.2) has a unique strict solution V. It follows that there exist maps $M, M_\alpha : \Omega \to \mathbb{R}$ such that all for $s,t \in [0,T]$

$$\begin{aligned}\|U(t) - U(s)\| &\leq \|G_W(t)V(t) - G_W(s)V(s)\| \\ &\leq \|G_W(t)V(t) - G_W(t)V(s)\| + \|G_W(t)V(s) - G_W(s)V(s)\| \\ &\leq M\|V(t) - V(s)\| + M_\alpha|t-s|^\alpha\|V(s)\|_{\mathcal{D}(C_W(0))}.\end{aligned}$$

The first term can be estimated because V is continuously differentiable. We already observed that $(C_W(s) - \mu)_{s\in[0,T]}$ satisfies (T2) for μ large. In particular $(C_W(0) - \mu)(C_W(s) - \mu)^{-1}$ is uniformly bounded in $s \in [0,T]$. Since $s \mapsto C_W(s)V(s)$ and V are uniformly bounded, we conclude that $\|V(s)\|_{\mathcal{D}(C_W(0))}$ is uniformly bounded. The result follows from this. □

6. Wong-Zakai approximations

As has been shown in [10] for a related class of problems in a Hilbert space setting, the techniques of this paper can be used to prove Wong-Zakai type approximation results for the problem (1.2),

$$\begin{aligned}dU(t) &= A(t)U(t)dt + BU(t)\,dW(t), \qquad t \in [0,T], \\ U(0) &= u_0.\end{aligned}$$

and possible generalizations for time-dependent operators $B(t)$. We shall briefly sketch the main idea and defer the details to a forthcoming publication.

Let $W^{(n)}$ be adapted processes with C^1 trajectories such that almost surely, $\lim_{n\to\infty} W_n = W$ uniformly on $[0,T]$ and consider the problem

$$\begin{aligned}dU_n(t) &= (A(t) - \frac{1}{2}B^2)U_n(t)dt + BU_n(t)\,dW_n(t), \qquad t \in [0,T], \\ U(0) &= u_0.\end{aligned} \tag{6.1}$$

This equation may be solved path by path as follows. Under the assumptions made in Section 3 and using the notations introduced there, define

$$C_{W_n}(t,\omega) := G^{-1}(W_n(t,\omega))C(t)G(W_n(t,\omega))$$

and consider the pathwise deterministic problem

$$
\begin{aligned}
V_n'(t) &= C_{W_n}(t)V_n(t), \quad t \in [0,T], \\
V_n(0) &= u_0.
\end{aligned} \tag{6.2}
$$

Arguing as in the proof of Theorem 3.1, $U_n := G(W_n)V_n$ is a strong solution of (6.1) if and only if V_n is a strong solution of (6.2), the difference being that instead of the Itô formula the ordinary chain rule is applied; this accounts for the loss of a factor $\frac{1}{2}B^2$.

In analogy to [10, Theorems 1 and 2], under suitable conditions on the operators $A(t)$ and B such as given in Sections 4 and 5 it can be shows that $\lim_{n\to\infty} V_n = V$ almost surely, where V is the strong solution of (3.2) and the almost sure convergence takes place in the functional space to which the trajectories of V belong. It follows that $\lim_{n\to\infty} U_n = U$ almost surely, where U is the strong solution of (1.2) and again the almost sure convergence takes place in the functional space to which the trajectories of U belong.

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